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Frames built on fractal sets

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Abstract

Frames of finite dimensional Hilbert spaces have recently been of great interest in applications to modern communication networks transport packets. In this note, continuous and discrete frames, living on fractal sets, of both finite and infinite dimensional separable abstract Hilbert spaces are found. In particular, we find discrete frames, robust to erasures, of finite dimensional Hilbert spaces using iterated function systems.

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1. Introduction

Hilbert spaces are the underlying mathematical structure of many areas of physical applications, predominantly, quantum theories and signal analysis. Signal analysis is concerned with decomposing and reconstructing vectors of a Hilbert space using simpler vectors without loosing essential information. What are these simpler vectors and how efficient are they? Since computers can only work with a finite set of data points, these essential components have to be collected with extra care. This practice may not be successful if we decompose the signal in terms of an orthonormal basis of the Hilbert space of the problem because in such a case the decomposition will be unique. The drawbacks of an orthonormal basis is argued well in [8,9]. In order to overcome this difficulty, practitioners looked for an alternative. As a result, by relaxing the orthogonality, overcomplete families of vectors were considered, such families are called frames. A specific type of frame is called wavelet [1,13].

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The theory of frames was initiated by Duffin and Schaeffer [10] in 1952 in the context of nonharmonic Fourier series. A significant role of frames in signal processing was pioneered by Daubechies et al. [8], and a landmark development was given by Daubechies [9]. Frames and their applications are very active areas of current research [4,6]. Frames have been used in signal processing because of their resilience to additive noise, their numerical stability of reconstruction [9], and greater freedom to capture signal characteristics [6]. For example, [11] uses the redundancy of a frame to mitigate the effect of losses in packet-based communication systems. Further, many of the applications of frames involve modeling and constructing infinite frames for infinite dimensional Hilbert spaces [4].

The theory of iterated function systems has found important applications in the area of computer graphics [3], and in modeling interference effects in quantum mechanics [5]. Recently, iterated function systems has been used to provide a new method of computing entropy for some classical and quantum dynamical systems [14]. The primary aim of this note is to find a link between two important theories, frames and iterated function systems and to develop their applications. In [15] we used iterations of rational functions to obtain frames parameterized by the elements of a Julia set. In this note, we use iterated function systems to obtain continuous and discrete frames, living on fractal sets, of both finite and infinite dimensional separable abstract Hilbert spaces.

In Section 2 we present definitions and the set up of the problem. In Section 3 we discuss continuous and discrete frames on infinite dimensional Hilbert spaces using iterations. Section 4 deals with a continuous frame on infinite dimensional Hilbert space, where we use probability functions together with the iterations to obtain frames. Sections 2–4 use fractals which are significantly away from the origin to obtain frames. Using a distance function, in Section 5, we obtain frames on fractals which contain the origin in it. In Section 6 we discuss continuous and discrete frames on finite dimensional Hilbert spaces. In Section 7, following the ideas developed in [6,11,12], we use the frames of this paper in applications to modern communication networks transport packets and compare our results to the results of [6,12].

2. Preliminaries and set up

We start with a general definition of a frame [1,2,7,13].

Definition 2.1. Let (X, μ) be a locally compact measure space and $\mathfrak{H}$ be an abstract separable Hilbert space. The family of vectors

$$\mathfrak{S} = \{\eta_x | x \in X\} \subset \mathfrak{H} \tag{2.1}$$

is said to form a frame in $\mathfrak{H}$ if the operator

$$F = \int_X |\eta_x\rangle\langle\eta_x| \, \mathrm{d}\mu \tag{2.2}$$

satisfies

$$A\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq B\|\phi\|^2 \quad \text{for all} \quad \phi \in \mathfrak{H}, \tag{2.3}$$

where A and B are positive constants. If $A = B$ the set $\mathfrak{S}$ is called a tight frame. If the operator $F = I$, the identity operator of $\mathfrak{H}$, then the set $\mathfrak{S}$ is said to give a resolution of the identity. Note that if $X = \mathbb{J}$ is a discrete set and μ is a counting measure, the operator (2.2) takes the form

$$F = \sum_{j\in\mathbb{J}} |\eta_j\rangle\langle\eta_j|. \tag{2.4}$$

In the case where X is partly discrete, the corresponding part of μ could in general be a weighted counting measure and (2.2) takes the form

$$F = \sum_{j\in\mathbb{J}'} \int_{X'} |\eta_{x,j}\rangle\langle\eta_{x,j}|\, \mathrm{d}\nu(x), \tag{2.5}$$

where $X = X' \cup \mathbb{J}'$, X' is the continuous part with measure ν and $\mathbb{J}'$ the discrete part with a counting measure on it.

We introduce some frame terminology [6]:

(i) Equal norm frame: $\|\eta_x\| = \|\eta_y\| \ \ \forall x, y \in X$.
(ii) Unit norm frame: $\|\eta_x\| = 1 \ \ \forall x \in X$.
(iii) Parseval tight frame: $A = B = 1$.

The quantity

$$w = \frac{B - A}{B + A} \tag{2.6}$$

is called the width of the frame. The width of a frame measures the tightness of the frame.

Now, we present the set up of the problem.

Let (X, d) be a complete metric space and $\tau_1, \tau_2, \ldots, \tau_K$ be a collection of transformations from X to itself. We assume that $\tau_1, \tau_2, \ldots, \tau_K$ are contractions, i.e., for $k = 1, \ldots, K$

$$\max_k d(\tau_k(x), \tau_k(y)) \le \alpha \cdot d(x, y),$$

where $\alpha < 1$. An iterated function system, IFS for short, with state dependent probabilities, $T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots, p_K(x)\}$ is defined by choosing $\tau_k(x)$ with probability $p_k(x)$, $p_k(x) > 0$, $\sum_{k=1}^K p_k(x) = 1$. The iterates of T are given by

$$T^m(x) = \tau_{k_m} \circ \tau_{k_{m-1}} \circ \cdots \circ \tau_{k_1}(x) \tag{2.7}$$

with probability

$$p_{k_m}(\tau_{k_{m-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{m-1}}(\tau_{k_{m-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x).$$

Let $(\mathcal{H}(X), h(d))$ denote the space of nonempty compact subsets corresponding to (X, d) with Hausdorff metric $h(d)$ [3]. $T : \mathcal{H}(X) \to \mathcal{H}(X)$ is given by

$$T(B) = \cup_{k=1}^K \tau_k(B)$$

for all $B \in \mathcal{H}(X)$. The Banach contraction theorem furnishes an attractor to the dynamical system (X, T) [3]. In the special case when $X = \mathbb{R}^2$, the attractor of an IFS is called a fractal

[3]. The presence of the probabilities allows more weight on some transformations over the others. Thus, the fractal may have parts more "dense" than the remaining parts.

Since we are on $\mathbb{R}^2$, the transformations are two component objects, i.e.,

$$T^m(x) = (f^m(x), g^m(x)).$$

From now on we interpret $T^m(x)$ as the complex number $f^m(x) + ig^m(x)$ and denote it by the same symbol, $T^m(x)$. We take a Hilbert space $\mathfrak{H}$ over complex numbers, thus the object $T^m(x)\phi$ is well-defined in $\mathfrak{H}$ for all $\phi \in \mathfrak{H}$.

Let $\mathbb{A}$ denote the attractor of the IFS and $\mathfrak{B}$ the Borel subsets of $(\mathbb{A}, d)$. Let μ be a probability measure on $\mathfrak{B}$ such that

$$\int_{\mathbb{A}} \mathrm{d}\mu(x) = 1. \tag{2.8}$$

In particular, one can use the probabilities to construct a measure on the attractor. For example, if the probabilities are constants, the measure can be constructed in the following way: $\mu(\mathbb{A}) = 1$, $\mu(\tau_k(\mathbb{A})) = p_k$, $\mu(\tau_l \circ \tau_k(\mathbb{A})) = p_l \cdot p_k$, and so on.

3. Frames on fractals using iterations

In this section, we assume that $X = \mathbb{R}^2$ and $d = |\cdot|$ is the Euclidean metric. We also assume that there exist a neighborhood, N_0, of the origin such that

$$N_0 \cap \mathbb{A} = \emptyset. \tag{3.1}$$

Remark 3.1. Observe that if T satisfies (3.1), we have

$$A \leq |T^m(x)| \leq B \quad \text{for all} \quad m = 0, 1, 2 \ldots \quad \text{and for all} \quad x \in \mathbb{A}, \tag{3.2}$$

where A and B are positive constants. The contraction property of the IFS, which grantees the existence of the attractor, is essential in the proof of our theorems. Infact, the contraction factor is the upper bound of the frame operator in study.

In the context of signal processing the interesting Hilbert space would be $\mathfrak{H} = L^2(\mathbb{A},\ \mathrm{d}\mu)$. The elements of this Hilbert space can be interpreted as finite energy signals while the measure space $(\mathbb{A},\ \mathrm{d}\mu)$ serve as the space of parameters. Since the following construction can be carried out with any separable Hilbert space, we take $\mathfrak{H}$ to be an abstract separable Hilbert space and $\{\phi_m\}_{m=0}^{\infty}$ is an orthonormal basis of it. If $\mathfrak{H}$ is a finite dimensional Hilbert space then the orthonormal basis takes the form $\{\phi_m\}_{m=1}^{N}$, where N is the dimension of the Hilbert space. Frames on finite dimensional Hilbert spaces are of practical interest in several directions, for example see [4,6,11,12].

In the following theorem we present a general construction of a continuous frame.

Theorem 3.2. *For $x \in \mathbb{A}$ let*

$$\phi_{x,m} = T^m(x)\phi_m, \tag{3.3}$$

and $S = \{\phi_{x,m} | x \in \mathbb{A}, m = 0, 1, 2, \ldots\}$. *If condition* (3.1) *is satisfied, the set S constitute a frame in* $\mathfrak{H}$. *That is, the operator*

$$F = \sum_{m=0}^{\infty} \int_{\mathbb{A}} |\phi_{x,m}\rangle\langle\phi_{x,m}| \, \mathrm{d}\mu(x)$$

satisfies

$$C\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq D\|\phi\|^2$$

for all $\phi \in \mathfrak{H}$ *and some positive constants C and D.*

Proof. We have

$$\langle\phi|F\phi\rangle = \sum_{m=0}^{\infty} \int_{\mathbb{A}} \langle\phi|\phi_{x,m}\rangle\langle\phi_{x,m}|\phi\rangle \, \mathrm{d}\mu(x) = \sum_{m=0}^{\infty} \int_{\mathbb{A}} \langle\phi|T^m(x)\phi_m\rangle\langle T^m(x)\phi_m|\phi\rangle \, \mathrm{d}\mu(x), \tag{3.4}$$

$$\langle\phi|F\phi\rangle = \sum_{m=0}^{\infty} \int_{\mathbb{A}} |T^m(x)|^2 \langle\phi|\phi_m\rangle\langle\phi_m|\phi\rangle \, \mathrm{d}\mu(x) = \sum_{m=0}^{\infty} \int_{\mathbb{A}} |T^m(x)|^2 |\langle\phi|\phi_m\rangle|^2 \, \mathrm{d}\mu(x). \tag{3.5}$$

By (2.8), (3.2) and (3.4) we obtain:

$$\langle\phi|F\phi\rangle \leq B^2 \sum_{m=0}^{\infty} |\langle\phi|\phi_m\rangle|^2 = B^2\|\phi\|^2$$

and

$$\langle\phi|F\phi\rangle \geq A^2 \sum_{m=0}^{\infty} |\langle\phi|\phi_m\rangle|^2 = A^2\|\phi\|^2.$$

This ends the proof. □

Remark 3.3. In the terminology of [2], the frame constructed in Theorem 3.2 can be named as a *continuous frame of infinite rank.*

In the context of signal processing, the selection $(\mathbb{A}, \mathrm{d}\mu, \{\psi_{z,m}\})$ reflects the selection of a part of the signal which we intend to isolate and analyze [1].

3.1. Discrete frame

In practice one has to deal with a discrete set of vectors, in fact, with a finite set of vectors. The discretization of a continuous process is often difficult and ill-posed [1,13]. Here we propose a method for discretizing the above constructed frame by choosing a finite set of points in the attractor.

Theorem 3.4. *For a fixed integer $N \in \mathbb{N}$ let $x_i \in \mathbb{A}$ and*

$$\phi_{x_i,m} = T^m(x_i)\phi_m, \tag{3.6}$$

and $S_N = \{\phi_{x_i,m} | x_i \in \mathbb{A}, i = 1, 2, \ldots, N; m = 0, 1, 2, \ldots\}$. If condition (3.1) is satisfied, the set S constitutes a frame in $\mathfrak{H}$. That is, the operator

$$F = \sum_{i=1}^{N}\sum_{m=0}^{\infty} |\phi_{x_i,m}\rangle\langle\phi_{x_i,m}|$$

satisfies

$$C\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq D\|\phi\|^2$$

for all $\phi \in \mathfrak{H}$ and some positive constants C and D.

Proof. We have

$$\begin{aligned}\langle\phi|F\phi\rangle &= \sum_{i=1}^{N}\sum_{m=0}^{\infty}\langle\phi|\phi_{x_i,m}\rangle\langle\phi_{x_i,m}|\phi\rangle = \sum_{i=1}^{N}\sum_{m=0}^{\infty}\langle\phi|T^m(x_i)\phi_m\rangle\langle T^m(x_i)\phi_m|\phi\rangle \\ &= \sum_{i=1}^{N}\sum_{m=0}^{\infty}|T^m(x_i)|^2\langle\phi|\phi_m\rangle\langle\phi_m|\phi\rangle = \sum_{i=1}^{N}\sum_{m=0}^{\infty}|T^m(x_i)|^2|\langle\phi|\phi_m\rangle|^2.\end{aligned} \tag{3.7}$$

By (3.2) and (3.7) we obtain:

$$\langle\phi|F\phi\rangle \leq NB^2\sum_{m=0}^{\infty}|\langle\phi|\phi_m\rangle|^2 = NB^2\|\phi\|^2$$

and

$$\langle\phi|F\phi\rangle \geq NA^2\sum_{m=0}^{\infty}|\langle\phi|\phi_m\rangle|^2 = NA^2\|\phi\|^2.$$

This ends the proof. □

We can also obtain a discrete frame with a discrete infinite set of points. We present it in the following theorem. It may be interesting to notice that the cardinality of the frame is the same as the cardinality of the orthonormal basis.

Theorem 3.5. *Let $x_m \in \mathbb{A}$ be a discrete set of points and*

$$\phi_{x_m,m} = T^m(x_m)\phi_m, \tag{3.8}$$

and $S = \{\phi_{x_m,m} | m = 0, 1, 2, \dots\}$. *If condition* (3.1) *is satisfied, the set S constitute a frame in* $\mathfrak{H}$. *That is, the operator*

$$F = \sum_{m=0}^{\infty} |\phi_{x_m,m}\rangle\langle\phi_{x_m,m}|$$

satisfies

$$C\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq D\|\phi\|^2$$

for all $\phi \in \mathfrak{H}$ *and some positive constants C and D*.

Proof. Proof is similar to the proof of Theorem 3.4. □

Corollary 3.6. *For each* $x \in \mathbb{A}$ *the set* $S_x = \{\phi_{x,m} | m = 0, 1, 2, \dots\}$ *is an orthogonal family in* $\mathfrak{H}$. *Further it is a frame*.

Remark 3.7. In the case where one intend to use a different label for the x_m of (3.8), the result of Theorem 3.5 fail to hold. That is

$$S = \{\phi_{x_j,m} | j = 1, 2, \dots ; m = 0, 1, 2, \dots\}$$

is not a frame because in this case the number N of Theorem 3.4 is infinity and thereby the frame bounds become infinite.

Often discretization of a continuous frame changes the frame width [13]. In our case, by Theorems 3.2 and 3.4 we can see that through the discretization process the frame width remains the same while the frame bounds change.

Now, we present an example which satisfies assumption (3.1) and hence Theorems 3.2, 3.4 and 3.5.

Example 3.8. Let $T = \{\tau_1, \tau_2, \tau_3\}$, $x = (t_1, t_2) \in \mathbb{R}^2$, where $\tau_1 = ((1/2)t_1, (1/2)t_2) + (1/2, 0)$, $\tau_2 = ((1/2)t_1, (1/2)t_2) + (1, 0)$ and $\tau_3 = ((1/2)t_1, (1/2)t_2) + (3/4, \sqrt{3}/4)$. Notice that T is a contraction with a contractivity factor 1/2. The attractor of T is shown in Fig. 1 with the origin outside the attractor. For this example, the vertices of the triangle are $(1, 0)$, $(2, 0)$ and $(3/2, \sqrt{3}/2)$. Thus, T satisfies condition (3.1).

4. Labeling with probability functions

In this section, we present another class of continuous frame using the probability functions together with the iterations. These probability functions bring additional weight on the frame vectors, and can be used to control the norm of the vectors. We use the assumptions of Section 3. Consider the set of vectors

$$\mathfrak{S} = \{\psi_{m,x,k} : x \in \mathbb{A}, m = 0, 1, 2, \dots ; k = 1, \dots, K\},$$

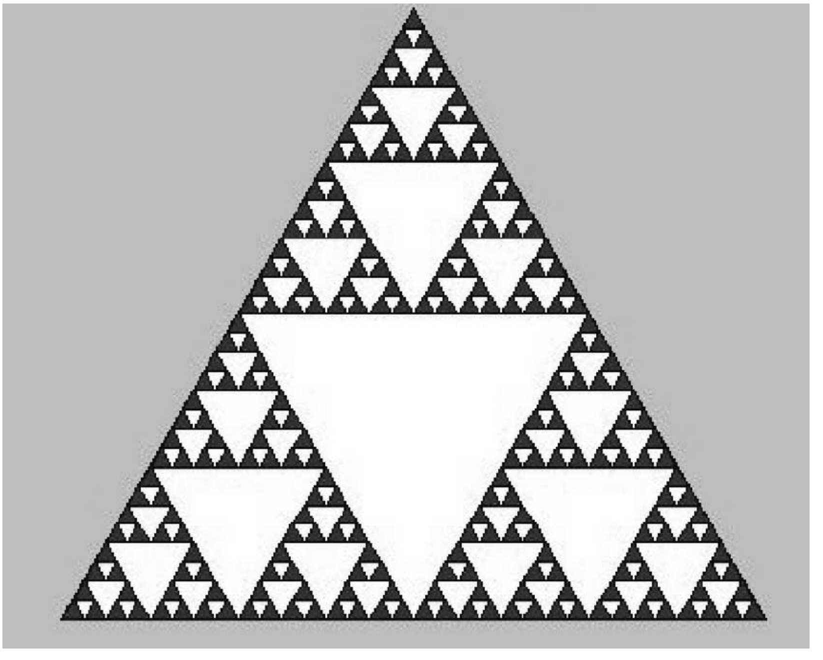

Fig. 1. Sierpinkski triangle, the attractor of the IFS in Examples 3.8 and 5.4.

where

$$\psi_{m,x,k} = (p_{k_m}(\tau_{k_{m-1}} \circ \cdots \circ \tau_{k_1}(x))\, p_{k_{m-1}} \times (\tau_{k_{m-2}} \circ \cdots \circ \tau_{k_1}(x)) \ldots p_{k_1}(x))^{1/2} \cdot T^m(x)\phi_m. \tag{4.1}$$

Theorem 4.1. *The set of vectors $\mathfrak{S}$ is a frame in $\mathfrak{H}$, that is the operator*

$$F = \sum_{k=1}^{K} \sum_{m=0}^{\infty} \int_{\mathbb{A}} |\psi_{m,x,k}\rangle\langle\psi_{m,x,k}|\, \mathrm{d}\mu(x) \tag{4.2}$$

satisfies the frame condition (2.3).

Proof. We have

$$\langle\phi|F\phi\rangle = \sum_{k=1}^{K} \sum_{m=0}^{\infty} \int_{\mathbb{A}} \langle\phi|\psi_{m,x,k}\rangle\langle\psi_{m,x,k}|\phi\rangle\, \mathrm{d}\mu(x)$$
$$= \sum_{k=1}^{K} \sum_{m=0}^{\infty} \int_{\mathbb{A}} \langle\phi|(p_{k_m}(\tau_{k_{m-1}} \circ \cdots \circ \tau_{k_1}(x)) \ldots p_{k_1}(x))^{1/2} \cdot T^m(x)\phi_m\rangle\langle(p_{k_m}(\tau_{k_{m-1}} \circ \cdots \circ \tau_{k_1}(x)) \ldots p_{k_1}(x))^{1/2} \cdot T^m(x)\phi_m|\phi\rangle\, \mathrm{d}\mu(x)$$

$$= \sum_{k=1}^{K}\sum_{m=0}^{\infty}\int_{\mathbb{A}} p_{k_m}(\tau_{k_{m-1}}\circ\cdots\circ\tau_{k_1}(x))\ldots p_{k_1}(x)\cdot |T^m(x)|^2\langle\phi|\phi_m\rangle\langle\phi_m|\phi\rangle\,\mathrm{d}\mu(x)$$
$$= \sum_{k=1}^{K}\sum_{m=0}^{\infty}\int_{\mathbb{A}} p_{k_m}(\tau_{k_{m-1}}\circ\cdots\circ\tau_{k_1}(x))\ldots p_{k_1}(x)\cdot |T^m(x)|^2|\langle\phi|\phi_m\rangle|^2\,\mathrm{d}\mu(x). \tag{4.3}$$

Therefore,

$$\langle\phi|F\phi\rangle \le B^2\sum_{m=0}^{\infty}\sum_{k=1}^{K}\int_{\mathbb{A}} p_{k_m}(\tau_{k_{m-1}}\circ\cdots\circ\tau_{k_1}(x))\ldots p_{k_1}(x)|\langle\phi|\phi_m\rangle|^2\,\mathrm{d}\mu(x)$$
$$= B^2\sum_{m=0}^{\infty}\int_{\mathbb{A}}|\langle\phi|\phi_m\rangle|^2\,\mathrm{d}\mu(x) = B^2||\phi||^2, \tag{4.4}$$

on the other hand

$$\langle\phi|F\phi\rangle \ge A^2\sum_{m=0}^{\infty}\sum_{k=1}^{K}\int_{\mathbb{A}} p_{k_m}(\tau_{k_{m-1}}\circ\cdots\circ\tau_{k_1}(x))\ldots p_{k_1}(x)|\langle\phi|\phi_m\rangle|^2\,\mathrm{d}\mu(x)$$
$$= A^2\sum_{m=0}^{\infty}\int_{\mathbb{A}}|\langle\phi|\phi_m\rangle|^2\,\mathrm{d}\mu(x) = A^2||\phi||^2. \tag{4.5}$$

The proof follows from the inequalities (4.4) and (4.5). □

Remark 4.2. The frame of Theorem 4.1 can be discretized by discretizing the continuous parameter x. As we did in Theorem 3.4, one can accomplish it by taking a discrete finite set of points $x_1, \ldots, x_N$ from the attractor.

5. Frames on fractals via distance

So far we have considered fractal sets which are significantly away from the origin. In the case where the origin is in the fractal set the above procedure cannot be applied. In this section we propose a way of having frames on such fractal sets. We use the set up of Section 2 and we assume that there exist a reference point $x_0 \in \mathbb{R}^2$ such that

$$N_{x_0} \cap \mathbb{A} = \emptyset, \tag{5.1}$$

where N_{x_0} is a neighborhood of x_0. Since T is a contraction

$$T^m(x) \in \mathbb{A} \quad \text{for all} \quad x \in \mathbb{A} \quad \text{and for all} \quad m = 0, 1, 2, \ldots, \tag{5.2}$$

we can fix $x_0 \in \mathbb{R}^2$ such that (5.1) is satisfied. Therefore, there exist a positive constant, A such that

$$A \leq \inf_{x\in\mathbb{A}, m=0,1,2,\ldots} d(T^m(x), x_0). \tag{5.3}$$

Since $\mathbb{A}$ is bounded, by (5.2) there exist a positive constant B such that

$$\sup_{x\in\mathbb{A}, m=0,1,2,\ldots} d(T^m(x), x_0) \leq B < \infty. \tag{5.4}$$

From (5.3) and (5.4) we have

$$A \leq d(T^m(x), x_0) \leq B \quad \text{for all} \quad x \in \mathbb{A} \quad \text{and} \quad m = 0, 1, 2, \ldots \tag{5.5}$$

Now, we intend to have frames as follows:

$$\psi_{x,m} = d(T^m(x), x_0)\phi_m. \tag{5.6}$$

Theorem 5.1. *The set*

$$\mathfrak{S} = \{\psi_{x,m} : x \in \mathbb{A}, m = 0, 1, 2, \ldots\}$$

is a frame in $\mathfrak{H}$

Proof. Let

$$F = \sum_{m=0}^{\infty} \int_{\mathbb{A}} |\psi_{x,m}\rangle\langle\psi_{x,m}| \,\mathrm{d}\mu(x).$$

For $\phi \in \mathfrak{H}$ consider

$$\begin{aligned}\langle\phi|F\phi\rangle &= \sum_{m=0}^{\infty} \int_{\mathbb{A}} \langle\phi|\psi_{x,m}\rangle\langle\psi_{x,m}|\phi\rangle \,\mathrm{d}\mu(x) \\ &= \sum_{m=0}^{\infty} \int_{\mathbb{A}} d(T^m(x), x_0)^2 \langle\phi|\phi_m\rangle\langle\phi_m|\phi\rangle \,\mathrm{d}\mu(x) \\ &= \sum_{m=0}^{\infty} \int_{\mathbb{A}} d(T^m(x), x_0)^2 |\langle\phi|\phi_m\rangle|^2 \,\mathrm{d}\mu(x).\end{aligned}$$

From (5.5) and (2.8) we get

$$\sum_{m=0}^{\infty} \int_{\mathbb{A}} d(T^m(x), x_0)^2 |\langle\phi|\phi_m\rangle|^2 \,\mathrm{d}\mu(x) \leq B^2 \sum_{m=0}^{\infty} |\langle\phi|\phi_m\rangle|^2 = B^2\|\phi\|^2. \tag{5.7}$$

Again by (5.5) and (2.8) we get

$$\sum_{m=0}^{\infty} \int_{\mathbb{A}} d(T^m(x), x_0)^2 |\langle\phi|\phi_m\rangle|^2 \,\mathrm{d}\mu(x) \geq A^2 \sum_{m=0}^{\infty} |\langle\phi|\phi_m\rangle|^2 = A^2\|\phi\|^2. \tag{5.8}$$

Combining (5.7) and (5.8) gives

$$A^2\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq B^2\|\phi\|^2.$$

Thus $\mathfrak{S}$ is a frame in $\mathfrak{H}$. □

Remark 5.2. Here again, the frame of Theorem 5.1 can be discretized by discretizing the continuous parameter x.

Remark 5.3. From (3.2) and (5.5) it is evident that the frame width

$$w = \frac{B^2 - A^2}{B^2 + A^2}$$

is solely depending on the width of the fractal. A good frame means $w \ll 1$. Thus, in order to get a good frame one needs to consider a narrow fractal.

Now, we present an example of an IFS, T, whose attractor satisfy condition (5.1).

Example 5.4. Let $T = \{\tau_1, \tau_2, \tau_3\}$, $x = (t_1, t_2) \in \mathbb{R}^2$, where $\tau_1 = ((1/2)t_1, (1/2)t_2)$, $\tau_2 = ((1/2)t_1, (1/2)t_2) + (1/2, 0)$ and $\tau_3 = ((1/2)t_1, (1/2)t_2) + (1/4, \sqrt{3}/4)$. Notice that T is a contraction with a contractivity factor 1/2. The attractor of T is shown in Fig. 1. We pick up an x_0 in the big hole. For this example, the vertices of the triangle are $(0, 0)$, $(1, 0)$ and $(1/2, \sqrt{3}/2)$. Thus, T satisfies condition (5.1).

Remark 5.5. The following remarks applies to all the frames constructed in this note. We demonstrate it with the frame constructed in Theorem 3.2:

(a) The frame operator F is self-adjoint and invertible [2,13]. Thus one can define another frame as

$$\psi_{x,m} = F^{-1/2}\phi_{x,m}. \tag{5.9}$$

In this case we get a resolution of the identity

$$I = \sum_{m=0}^{\infty} \int_{\mathbb{A}} |\psi_{x,m}\rangle\langle\psi_{x,m}|\, \mathrm{d}\mu. \tag{5.10}$$

From (5.10) one can get a perfect reconstruction. However, explicit knowledge of $F^{-1/2}$ is of practical interest to use (5.10).

(b) One can define several equivalent frames associated to the constructed frame following the procedures given in [2,6,7,13]. We point out some briefly here. If one defines $\psi_{x,m} = F^{-1}\phi_{x,m}$ then the collection $\{\psi_{x,m} : x \in \mathbb{A}, m = 0, 1, \dots\}$ is a frame with frame operator F^{-1}. This frame is called the dual frame of the original frame.

For any operator $U \in GL(\mathfrak{H})$ if we define $\psi_{x,m} = U\phi_{x,m}$ then the collection $\{\psi_{x,m} : x \in \mathbb{A}, m = 0, 1, \dots\}$ is a frame with frame operator UFU^*, where U^* is the adjoint of U. If $UU^* = U^*U = I$ then the frame is said to be unitarily equivalent to the original frame.

6. Frames on finite dimensional Hilbert spaces

Frames on finite dimensional Hilbert spaces are of practical interest in many applications [4,6,11,12]. In this section we build frames on finite dimensional Hilbert spaces parameterized by the elements of a fractal. Here again we work on fractals those satisfy condition (3.1). Suppose $\mathfrak{H}$ is a finite dimensional Hilbert space and $\dim \mathfrak{H} = N$. Let $\{\phi_m\}_{m=1}^{N}$ is an orthonormal basis of $\mathfrak{H}$. Let $\zeta \in [0, 2\pi)$, $\mathrm{d}\zeta$ is the invariant measure on $[0, 2\pi)$ and

$$\tilde{T}^m(x) = \mathrm{e}^{\mathrm{i}m\zeta} T^m(x),$$

where $T^m(x)$ is as in (2.7). For $x \in \mathbb{A}$, let

$$\phi_{x,\zeta} = \sum_{m=1}^{N} \tilde{T}^m(x)\phi_m. \tag{6.1}$$

Theorem 6.1. *The set of vectors $S = \{\phi_{x,\zeta} : x \in \mathbb{A}, \zeta \in [0, 2\pi)\}$ is a frame in $\mathfrak{H}$.*

Proof. The frame operator takes the form

$$F = \int_0^{2\pi} \int_{\mathbb{A}} |\phi_{x,\zeta}\rangle\langle\phi_{x,\zeta}| \,\mathrm{d}\mu\, \mathrm{d}\zeta.$$

For $\phi \in \mathfrak{H}$ consider

$$\begin{aligned}
\langle\phi|F\phi\rangle &= \sum_{m=1}^{N}\sum_{k=1}^{N} \int_0^{2\pi} \int_{\mathbb{A}} \langle\phi|\tilde{T}^m(x)\phi_m\rangle\langle\phi|\tilde{T}^k(x)\phi_k\rangle \,\mathrm{d}\mu\, \mathrm{d}\zeta \\
&= \sum_{m=1}^{N}\sum_{k=1}^{N} \int_0^{2\pi} \int_{\mathbb{A}} \mathrm{e}^{\mathrm{i}(m-k)\theta}\langle\phi|T^m(x)\phi_m\rangle\langle\phi|T^k(x)\phi_k\rangle \,\mathrm{d}\mu\, \mathrm{d}\zeta \\
&= 2\pi \sum_{m=1}^{N} \int_{\mathbb{A}} \langle\phi|T^m(x)\phi_m\rangle\langle\phi|T^m(x)\phi_m\rangle \,\mathrm{d}\mu \\
&= 2\pi \sum_{m=1}^{N} \int_{\mathbb{A}} |T^m(x)|^2 |\langle\phi|\phi_m\rangle|^2 \,\mathrm{d}\mu.
\end{aligned}$$

Now from (2.8) and (3.2) we have

$$2\pi \sum_{m=1}^{N} \int_{\mathbb{A}} |T^m(x)|^2 |\langle\phi|\phi_m\rangle|^2 \,\mathrm{d}\mu \le 2\pi B^2 \sum_{m=1}^{N} |\langle\phi|\phi_m\rangle|^2 = 2\pi B^2 \|\phi\|^2.$$

Again from (2.8) and (3.2) we have

$$2\pi \sum_{m=1}^{N} \int_{\mathbb{A}} |T^m(x)|^2 |\langle\phi|\phi_m\rangle|^2 \,\mathrm{d}\mu \ge 2\pi A^2 \sum_{m=1}^{N} |\langle\phi|\phi_m\rangle|^2 = 2\pi A^2 \|\phi\|^2.$$

Thus the set S is a frame. □

Remark 6.2. The above theorem produces a continuous frame in a finite dimensional Hilbert space. The additional term $e^{i\zeta}$ is introduced for convenience. One can observe that without this term the above construction cannot be carried out. Further, the new variable ζ brings additional redundancy to the frame.

Equal norm frames are of practical interest [6]. In particular, equal norm Parseval tight frames are useful in the reconstruction process because they produce a resolution of the identity. In the case of a discrete frame, unit norm Parseval tight frames are, in fact, orthonormal bases. However, in the continuous case this is not true. We present these aspects, as a consequence of Theorem 6.1, in the following two propositions.

Let

$$\lambda(x) = \sum_{m=1}^{N} |T^m(x)|^2. \tag{6.2}$$

Now let us define a new set of vectors as

$$\psi_{x,\zeta} = \lambda(x)^{-1/2} \sum_{m=1}^{N} \tilde{T}^m(x)\phi_m, \tag{6.3}$$

and set a new measure $d\nu = \lambda(x)\, d\mu$ on $\mathbb{A}$.

Proposition 6.3. *The set of vectors* $S = \{\psi_{x,\zeta} : x \in \mathbb{A}, \zeta \in [0, 2\pi)\}$ *is a unit norm frame in* $\mathfrak{H}$.

Proof. One can easily see that $\langle \psi_{x,\zeta} | \psi_{x,\zeta} \rangle = 1$ for all $x \in \mathbb{A}, \zeta \in [0, 2\pi)$. The rest of the proof follows from Theorem 6.1 with the measure $d\nu$. □

Proposition 6.4. *Let* $\{\psi_{x,\zeta} : x \in \mathbb{A}, \zeta \in [0, 2\pi)\}$ *is as in* Proposition 6.3 *and F be the corresponding frame operator then* $\{\eta_{x,\zeta} = F^{-1/2}\phi_{x,\zeta} : x \in \mathbb{A}, \zeta \in [0, 2\pi)\}$ *is an equal norm Parseval tight frame in* $\mathfrak{H}$.

Proof. A proof follows from Remark 5.5(a) and Proposition 6.3. □

Remark 6.5. In quantum mechanical terminology the frames of Proposition 6.4 can be phrased as *coherent states* in the finite dimensional Hilbert space $\mathfrak{H}$. For definitions and details of coherent states [1] is an excellent reference.

As a first step to a complete discretization, we label frames by a partly discrete set. For this, let us discretize one label by picking an infinite number of discrete points from the fractal corresponding to T, and write a vector as follows:

$$\phi_{x_j,\zeta} = \sum_{m=1}^{N} \frac{1}{\sqrt{j!}} \tilde{T}^m(x_j)\phi_m. \tag{6.4}$$

Proposition 6.6. *The set* $S = \{\phi_{x_j,\zeta} : j = 0, 1, \dots ; \zeta \in [0, 2\pi)\}$ *is a frame in* $\mathfrak{H}$.

Proof. The frame operator takes the form

$$F=\sum_{j=0}^{\infty}\int_0^{2\pi}|\phi_{x_j,\zeta}\rangle\langle\phi_{x_j,\zeta}|\,\mathrm{d}\zeta.$$

Thus for $\phi\in\mathfrak{H}$ we have

$$\langle\phi|F\phi\rangle=\sum_{j=0}^{\infty}\sum_{m=1}^{N}\sum_{k=1}^{N}\frac{1}{\sqrt{j!k!}}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(m-k)\zeta}T^m(x_j)\overline{T^k(x_j)}\langle\phi|\phi_m\rangle\langle\phi_k|\phi\rangle\,\mathrm{d}\zeta$$
$$=\sum_{j=0}^{\infty}\sum_{m=1}^{N}\frac{2\pi}{j!}|T^m(x_j)|^2|\langle\phi|\phi_m\rangle|.$$

Now from (3.2) we have

$$\sum_{j=0}^{\infty}\sum_{m=1}^{N}\frac{2\pi}{j!}|T^m(x_j)|^2|\langle\phi|\phi_m\rangle|$$
$$\le\sum_{j=0}^{\infty}\frac{2\pi B^2}{j!}\sum_{m=1}^{N}|\langle\phi|\phi_m\rangle|=\sum_{j=0}^{\infty}\frac{2\pi B^2}{j!}\|\phi\|^2=2e\pi B^2\|\phi\|^2.$$

Again by (3.2) we have

$$\sum_{j=0}^{\infty}\sum_{m=1}^{N}\frac{2\pi}{j!}|T^m(x_j)|^2|\langle\phi|\phi_m\rangle|$$
$$\ge\sum_{j=0}^{\infty}\frac{2\pi A^2}{j!}\sum_{m=1}^{N}|\langle\phi|\phi_m\rangle|=\sum_{j=0}^{\infty}\frac{2\pi A^2}{j!}\|\phi\|^2=2e\pi A^2\|\phi\|^2.$$

Thus the set S forms a frame in $\mathfrak{H}$. □

One could restrict the infinite set of points to a finite set of points. We present it in the following corollary.

Corollary 6.7. *The set* $S_M=\{\phi_{x_j,\zeta}:j=0,1,\ldots,M,\zeta\in[0,2\pi)\}$ *is a frame in* $\mathfrak{H}$ *for each positive integer M.*

Proof. The proof is similar to the Proposition 6.6 with different frame bounds, i.e.,

$$\frac{2e\pi\Gamma(M+1,1)A^2}{M!}\|\phi\|^2\le\langle\phi|F\phi\rangle\le\frac{2e\pi\Gamma(M+1,1)B^2}{M!}\|\phi\|^2,$$

where $\Gamma(M+1,1)$ is the incomplete gamma. □

In the case of a finite set of points, in the frame operator, the sum representing the finite part is a finite sum thus the denominator $\sqrt{j!}$ is not necessary anymore (from the proof of Proposition 6.6 one can notice that the term is necessary to guarantee the convergence of the series). We present it without the term $\sqrt{j!}$ in the following corollary. However, from Corollaries 6.7 and 6.8, one can notice that the presence of $\sqrt{j!}$ significantly affect the frame bounds.

Corollary 6.8. *Let*

$$\psi_{x_j,\zeta} = \sum_{m=1}^{N} \tilde{T}^m(x_j)\phi_m. \tag{6.5}$$

The set $S_M = \{\psi_{x_j,\zeta} : j = 1, \dots, M, \zeta \in [0, 2\pi)\}$ *is a frame in* $\mathfrak{H}$ *for each finite positive integer* M.

Proof. Here again proof follows similar to the Proposition 6.6 with different frame bounds, i.e.,

$$2\pi MA^2\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq 2\pi MB^2\|\phi\|^2. \qquad \square$$

Remark 6.9. The frame obtained in Theorem 6.1 cannot be extended to infinite dimensional Hilbert spaces because in such a case $N = \infty$, and, thereby, the norm of the vector in (6.1) is infinity for each pair $(x, \zeta) \in \mathbb{A} \times [0, 2\pi)$.

Now we construct another class of frames on finite dimensional Hilbert spaces by following the procedure applied to infinite dimensional Hilbert spaces.

Theorem 6.10. *The set* $S = \{\eta_{x,m} = T^m(x)\phi_m : x \in \mathbb{A}, m = 1, 2, \dots, N\}$ *is a frame in* $\mathfrak{H}$.

Proof. A proof follows similar to Theorem 3.2. $\square$

6.1. Discrete frames on finite dimensional Hilbert spaces

In this section we build discrete frames on finite dimensional Hilbert spaces. Let $\dim(\mathfrak{H}) = N$. Frames on finite dimensional Hilbert spaces with finite number of elements are used in several applications [4,6,11,12]. We construct a discrete frame with MN elements in the following theorem.

Theorem 6.11. *The set* $S = \{\eta_{x_j,m} = T^m(x_j)\phi_m : j = 1, \dots, M; m = 1, \dots, N\}$ *is a frame in* $\mathfrak{H}$.

Proof. The frame operator takes the form

$$F = \sum_{j=1}^{M}\sum_{m=1}^{N} |\eta_{x_j,m}\rangle\langle\eta_{x_j,m}|.$$

One can easily see that

$$A^2 M\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq B^2 M\|\phi\|^2. \qquad \square$$

From the frame of the Theorem 6.11 one can obtain a unit norm M-tight frame as follows.

Corollary 6.12. *Let*

$$\bar{\eta}_{x_j,m} = \frac{T^m(x_j)}{|T^m(x_j)|}\phi_m.$$

The set $S = \{\bar{\eta}_{x_j,m} : j = 1, \ldots, M; m = 1, \ldots, N\}$ is a unit norm M-tight frame with MN elements.

Proof. One can easily see that $\|\bar{\eta}_{x_j,m}\| = 1$ and the frame operator, $F = MI_{\mathfrak{H}}$. $\square$

In the literature of frames, several classes of frames have been obtained by taking linear superpositions of the basis vectors of Hilbert spaces [1,6]. In the following two theorems we attempt to adapt it to our construction. For this, let us take a mutually disjoint partition, $\{A_m : m = 1, \ldots, N\}$ of a fractal $\mathbb{A}$. That is,

$$\mathbb{A} = \bigcup_{m=1}^{N} A_m,\ A_m \cap A_n = \emptyset, \quad \forall m \neq n.$$

Let $\chi_{A_m}(x)$ be the characteristic function of A_m.

Theorem 6.13. *Let $M \geq N$ and $\{x_1, \ldots, x_M\} \subset \mathbb{A}$ such that $\{x_1, \ldots, x_M\} \cap A_m \neq \emptyset$, for all $m = 1, \ldots, N$. Define*

$$\psi_j = \sum_{m=1}^{N} T^m(x_j)\chi_{A_m}(x_j)\phi_m. \tag{6.6}$$

The set $S_M = \{\psi_j : j = 1, \ldots, M\}$ is a frame in $\mathfrak{H}$.

Proof. The frame operator takes the form

$$F = \sum_{j=1}^{M} |\psi_j\rangle\langle\psi_j|.$$

Thus, for any $\phi \in \mathfrak{H}$

$$\begin{aligned}\langle\phi|F\phi\rangle &= \sum_{j=1}^{M}\sum_{m=1}^{N}\sum_{l=1}^{N} T^m(x_j)\chi_{A_m}(x_j)\overline{T^l(x_j)}\chi_{A_l}(x_j)\langle\phi|\phi_m\rangle\langle\phi_l|\phi\rangle \\ &= \sum_{j=1}^{M}\sum_{m=1}^{N} |T^m(x_j)|^2\chi_{A_m}(x_j)|\langle\phi|\phi_m\rangle|^2\end{aligned}$$

because $A_m \cap A_n = \emptyset, \quad \forall m \neq n$ implies

$$\chi_{A_m}(x_j)\chi_{A_l}(x_j) = \begin{cases} 0 & \text{if } m \neq l, \\ \chi_{A_m}(x_j) & \text{if } m = l. \end{cases}$$

By the definition of the characteristic function and by (3.2) we have

$$\sum_{j=1}^{M}\sum_{m=1}^{N} |T^m(x_j)|^2 \chi_{A_m}(x_j) |\langle\phi|\phi_m\rangle|^2 \leq B^2 \sum_{j=1}^{M}\sum_{m=1}^{N} |\langle\phi|\phi_m\rangle|^2 = B^2 M \|\phi\|^2.$$

Since $\{x_1, \ldots, x_M\} \cap A_m \neq \emptyset$, for all $m = 1, \ldots, N$, by (3.2) we have

$$\sum_{j=1}^{M}\sum_{m=1}^{N} |T^m(x_j)|^2 \chi_{A_m}(x_j) |\langle\phi|\phi_m\rangle|^2 \geq A^2 \sum_{j=1}^{M}\sum_{m=1}^{N} \chi_{A_m}(x_j) |\langle\phi|\phi_m\rangle|^2$$

$$\geq A^2 \sum_{m=1}^{N} |\langle\phi|\phi_m\rangle|^2 = A^2 \|\phi\|^2.$$

Thus

$$A^2\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq B^2 M \|\phi\|^2. \qquad \square$$

Remark 6.14. The vectors in (6.6) are only labeled by j. Since $A_m \cap A_n = \emptyset$ for all $m \neq n$, for each j, $x_j \in A_{m_j}$ for only one $m_j \in \{1, 2, \ldots, N\}$, thus

$$\psi_j = T^{m_j}(x_j)\phi_{m_j}.$$

Compare to Theorem 6.11, in Theorem 6.13 the label m is hidden by the sum.

In the next theorem, let us see how can we make the sum effective on the vector.

Theorem 6.15. *Let $M, L \in \mathbb{N}$ such that $L \geq N$. Let $\{x_1, \ldots, x_M\}$ and $\{y_1, \ldots, y_L\}$ are two discrete subsets of $\mathbb{A}$ such that $\{y_1, \ldots, y_L\} \cap A_m \neq \emptyset$ for all $m = 1, \ldots, N$. Define*

$$\psi_k = \sum_{j=1}^{M}\sum_{m=1}^{N} T^m(x_j)\chi_{A_m}(y_k)\phi_m.$$

The set $S_L = \{\psi_k : k = 1, \ldots, L\}$ is a frame in $\mathfrak{H}$.

Proof. A proof follows by the same techniques of Theorem 6.13. In this case we get

$$A^2 M\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq B^2 ML \|\phi\|^2. \qquad \square$$

Remark 6.16. Even though for each k the vector ψ_k depends on the sum, ψ_k depends only one basis element, because for each k, $y_k \in A_{m_k}$ for only one $m_k \in \{1, \ldots, N\}$. Thus the vector is, in effect

$$\psi_k = \sum_{j=1}^{M} T^{m_k}(x_j)\phi_{m_k}.$$

7. Communication networks

Frames are of central interest in signal analysis. Recently, frames of finite dimensional Hilbert spaces were applied in the study of modern communication networks transport packets of data from a source to a recipient [11,12]. In particular, equal norm tight frames are very useful in such analysis at the presence of one or more erased information [4,6,12].

The frames constructed in this note can be adapted to the above applications. Moreover, we have generated equal norm tight frames at some instances. Our discrete frames are flexible through the points selected in the fractals, the IFS T, and the probability functions. It may also be interesting to notice that for most of the frames constructed in this note, the width of the frame is determined by the size (width) of the fractal.

Now, we compare our results to those of [6,12].

A modern communication network, for example Internet, provides means to transport packets of data from one device to another. In this process at some point the system experiences losses [6,12]. Abstraction of this process can be done with frame expansion (for more details see Fig. 1 of [12]). A signal vector $\phi \in \mathfrak{H}$ is expanded with a frame operator F and the frame coefficients are transmitted as packets of data. The lost packets at intermediate nodes of the network are modeled as erasures of transmitted frame coefficients. At the receiver side, this looks like the original frame without vectors corresponding to erased coefficients. The question is: Can the receiver recover the original information without errors among losses? If the lost packet is independent of the other transmitted data, then the information is truly lost to the receiver (see Section 5 of [6] or the Example of page 219 of [12]). If there are dependencies between transmitted packets one could have partial or complete recovery despite losses.

We start our comparison with the notion of a frame robust to k-erasures or a k-robust frame [6]. In the following discussion $\mathfrak{H}$ is an N-dimensional Hilbert space.

Definition 7.1. A frame $\{\phi_i\}_{i=1}^M$ of $\mathfrak{H}$ is said to be robust to k-erasures if $\{\phi_i\}_{i\in J\setminus I}$ is still a frame for I any indexed subset of k-erasures, $I \subset \{1, 2, \ldots, M\} = J$, $\mathrm{card}(I) = k$.

Theorem 7.2. *Let $J = \{1, \ldots, M\}$, $K = \{1, \ldots, N\}$ and S be the frame of* Theorem 6.11. *The frame S is robust to* card(I)*-erasures for any $I \subseteq J$.*

Proof. Let $L = \mathrm{card}(I) \leq M$, then one can see that the frame operator

$$F = \sum_{j\in J\setminus I} \sum_{m=1}^{N} |\eta_{x_j,m}\rangle\langle\eta_{x_j,m}|$$

satisfies

$$A^2(M-L)\|\phi\|^2 \le \langle\phi|F\phi\rangle \le B^2(M-L)\|\phi\|^2$$

for all $\phi \in \mathfrak{H}$. □

The frame S of Corollary 6.12 provides us with a unit norm M-tight frame with MN elements. In fact, Theorems 6.11, 6.13 and 6.15 and Corollary 6.12 provide constructive methods via the fractal sets to build discrete frames robust to erasures. Theorem 3.1 of [12] shows us that unit norm tight frames optimize robustness to quantization noise, accordingly, the frame S of Corollary 6.12 optimizes robustness to quantization noise. Further, Theorem 4.1 of [12] proves that equal norm M/N-tight frames with M elements of the Hilbert space $\mathbb{R}^N$ is robust to one-erasure with lower frame bound $A_1 = M/N - 1$ and upper frame bound $B_1 = M/N$. One can observe that, in this case, the frame looses its tightness.

In the following theorem, we are going to show that the frame S of Corollary 6.12 is robust to erasures up to certain degree without loosing its tightness.

Theorem 7.3. *Let $J = \{1, \dots, M\}$, $K = \{1, \dots, N\}$ and S be the frame of* Corollary 6.12. *The frame S is robust to card(I)-erasures for any $I \subseteq J$ and*

$$S_e = \{\bar{\eta}_{x_j,m} : j \in \{1, \dots, M\}\backslash I;\, m = 1, \dots, N\}$$

is $[M - \mathrm{card}(I)]$*-tight.*

Proof. Let $\mathrm{card}(I) = L$. It is straight forward to see that the frame operator

$$F = \sum_{j\in J\backslash I} \sum_{m=1}^{N} |\bar{\eta}_{x_j,m}\rangle\langle\bar{\eta}_{x_j,m}|$$

satisfies

$$\langle\phi|F\phi\rangle = (M-L)\|\phi\|^2, \quad \forall\phi \in \mathfrak{H}.$$ □

Remark 7.4. In agreement with the arguments of [6,12], the frames of Theorem 6.11 and Corollary 6.12 are not robust to erasures through the entries of the index m because, in such a case, a lost component is independent of the other components. For example, for the frame of Theorem 6.11, for any $I \subseteq \{1, \dots, M\}$

$$S_{e_1} = \{\eta_{x_j,m} : j \in \{1, \dots, M\}\backslash I;\, m = 1, \dots, N\}$$

is a frame, but for any $I \subseteq \{1, \dots, N\}$

$$S_{e_2} = \{\eta_{x_j,m} : j \in \{1, \dots, M\};\, m \in \{1, \dots, N\}\backslash I\}$$

is not a frame.

In the case of Theorem 4.1 of [12] more than one deletion from the frame is not discussed. Further, it was argued that unit norm tight frames may fail to remain a frame after $(M -$

N)-erasures. Theorem 4.2 of [12] proves that a class of harmonic frames are robust to $(M - N)$-erasures.

In the following, we show that the frames of Theorems 6.13 and 6.15 have similar properties. Moreover, one can see that the frames of Theorems 6.13 and 6.15 are built through fractal sets to any abstract Hilbert space of dimension N while the frames of Theorem 4.2 of [12] are a particular type of frames of $\mathbb{R}^N$ and $\mathbb{C}^N$.

Theorem 7.5. *Let S_M be the frame of Theorem 6.13. S_M is robust to* card(I)*-erasures if* card$(I) \leq M - N$ *and* $\{x_1, \ldots, x_{M-N}\} \cap A_m \neq \emptyset$ *for all* $m = 1, \ldots, N$.

Proof. Let card$(I) = q$ and $J = \{1, \ldots, M\}$, then following the proof of Theorem 6.13 we can see that the frame operator

$$F = \sum_{j \in J \setminus I} |\psi_j\rangle\langle\psi_j|$$

satisfies the frame condition

$$A^2\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq B^2(M - q)\|\phi\|^2, \quad \forall\phi \in \mathfrak{H}.$$

□

Theorem 7.6. *Let S_L be the frame of Theorem 6.15. S_L is robust to* card(I)*-erasures if* card$(I) \leq L - N$ *and* $\{y_1, \ldots, y_{L-N}\} \cap A_m \neq \emptyset$ *for all* $m = 1, \ldots, N$.

Proof. A proof follows similar to Theorem 6.13. □

References

[1] S.T. Ali, J.P. Antoine, J.P. Gazeau, Coherent States, Wavelets and their Generalizations, Springer, New York, 2000.
[2] S.T. Ali, J.P. Antoine, J.P. Gazeau, Continuous frames in Hilbert spaces, Ann. Phys. 222 (1993) 1–37.
[3] M. Barnsley, Fractals Everywhere, Academic Press, London, 1988.
[4] J.J. Benedetto, M. Fickus, Finite normalized tight frames, Adv. Comput. Math. 18 (2003) 357–385.
[5] A. Boyarsky, P. Góra, A dynamical system model for interference effects and the two-slit experiment of quantum physics, Phys. Lett. A 168 (1992) 103–112.
[6] P.G. Casazza, J. Kovačević, Equal-norm tight frames with erasures, Adv. Comput. Math. 18 (2003) 387–430.
[7] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.
[8] I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 (1986) 1271–1283.
[9] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, PA, 1992.
[10] R.J. Duffin, A.C. Schaeffer, A class of nonharmonic Fourier series, Trans. Am. Math. Soc. 72 (1952) 341–366.
[11] V.K. Goyal, J. Kovačević, M. Vetterli, Quantized frame expansions as source-channel codes for erasure channels, in: Proceedings of the Data Compr. Conference, Snowbird, UT, 1999, pp. 388–397.
[12] V.K. Goyal, J. Kovačević, J. Kelner, Quantized frame expansions with erasures, Appl. Comput. Harmon. Anal. 10 (2001) 203–233.
[13] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, MA, 2001.
[14] W. Somczyńskia, J. Kwapień, K. Życzkowskic, Entropy computing via integration over fractal measures, Chaos 10 (2000) 180–188.
[15] K. Thirulogasanthar, W. Bahsoun, Continuous and discrete frames on Julia sets, J. Geom. Phys., in press.